Mathematical Modelling and Analysis I

Coursework 1

Model 1: Elasticity

a) Given equation E.1 → $F = k\Delta x$
To find the dimensions of k we will use the dimensions of F and Δx
F is force
According to Newton's Second Law of Motion F = m.a
(where m = mass of body, a = acceleration of body)
Also, acceleration is speed per unit time and speed is distance travelled per unit time.
Thus, dimensions of a= $L T^{-1} / T = LT^{-2}$
(where L=dimension for Length and T=dimension for Time)
So, Dimensions of F = M . $LT^{-2} = MLT^{-2}$
(where M=dimension for mass)
Since Δx is displacement, dimensions of Δx = L

$F = k\Delta x$
⇨ $k = F/\Delta x$
⇨ Dimensions of k = (Dimensions of F)/(Dimensions of Δx)
⇨ Dimensions of k = (MLT^{-2}/L)
⇨ **Dimensions of k = MT^{-2} or Mass Per Time Squared**

b) Start by making and assumption
Assume that $k_1 = k_2 = k_3 = k$ (say) .....(1)
And that $l_1 = l_2 = l_3 = l$ (say) .....(2)
Also, since $L = l_1 + l_2 + l_3$
⇨ $L = 3l$ .....(3)
Now, given $T_1 = T_2$
⇨ $k_1(x_1 – l_1) = k_2(x_2 – x_1 – l_2)$
now using (1) and (2)
⇨ $k(x_1 – l) = k(x_2 – x_1 – l)$
⇨ $kx_1 – \cancel{kl} = kx_2 – kx_1 – \cancel{kl}$
⇨ $kx_1 + kx_1 – kx_2 = 0$
⇨ $2kx_1 – kx_2 = 0$
⇨ $k(2x_1 – x_2) = 0$
⇨ $\mathbf{2x_1 – x_2 = 0}$ .....(A)
And, given $T_2 = T_3$
⇨ $k_2(x_2 – x_1 – l_2) = k_3(L – x_2 – l_3)$
now using (1), (2), and (3)
⇨ $k(x_2 – x_1 – l) = k(3l – x_2 – l)$
⇨ $kx_2 – kx_1 – \cancel{kl} = 3kl – kx_2 – \cancel{kl}$
⇨ $kx_2 + kx_2 – kx_1 – 3kl = 0$

⇨ $2kx_2 - kx_1 = 3kl$

⇨ $k(2x_2 - x_1) = 3kl$

⇨ $\mathbf{2x_2 - x_1 = 3l}$ .....(B)

Equations (A) and (B) can be written in the form of a matrix as follow

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = l\begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Taking l outside to enable calculation via Matlab

Mat_A Mat_B Mat_C

Using Matlab we solve for Mat_B (refer to Appendix 1, Page 18)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = l\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

⇨ $x_1 = l$ and $x_2 = 2l$

using (3)

⇨ **$x_1 = L/3$ and $x_2 = 2L/3$**

c) Observing T_1, T_2, and T_3, I deduced that $T_n = k_n(x_n - x_{n-1} - l_n)$

So $\mathbf{T_1} = k_1(x_1 - x_0 - l_1) = \mathbf{k_1(x_1 - l_1)}$

$\mathbf{T_2 = k_2(x_2 - x_1 - l_2)}$

$\mathbf{T_3 = k_3(x_3 - x_2 - l_3)}$

$\mathbf{T_4} = k_4(x_4 - x_3 - l_4) = \mathbf{k_4(L - x_3 - l_4)}$ [because $L = x_4$]

Now making an assumption

Assume that $k_1 = k_2 = k_3 = k_4 = k$ (say) .....(1)

And that $l_1 = l_2 = l_3 = l_4 = l$ (say) .....(2)

Also, since $L = l_1 + l_2 + l_3 + l_4$

⇨ $L = 4l$ .....(3)

Now, given $T_1 = T_2$

⇨ $k_1(x_1 - l_1) = k_2(x_2 - x_1 - l_2)$

now using (1) and (2)

⇨ $k(x_1 - l) = k(x_2 - x_1 - l)$

⇨ $kx_1 - kl = kx_2 - kx_1 - kl$

⇨ $kx_1 + kx_1 - kx_2 = 0$

⇨ $2kx_1 - kx_2 = 0$

⇨ $k(2x_1 - x_2) = 0$

⇨ $\mathbf{2x_1 - x_2 + 0x_3 = 0}$ .....(A)

Also, given $T_2 = T_3$

⇨ $k_2(x_2 - x_1 - l_2) = k_3(x_3 - x_2 - l_3)$

now using (1) and (2)

⇨ $k(x_2 - x_1 - l) = k(x_3 - x_2 - l)$

⇨ $kx_2 - kx_1 - kl = kx_3 - kx_2 - kl$

⇨ $kx_2 - kx_1 - kx_3 + kx_2 = 0$

⇨ $-kx_1 + 2kx_2 - kx_3 = 0$

⇨ $k(-x_1 + 2x_2 - x_3) = 0$

⇨ $\mathbf{-x_1 + 2x_2 - x_3 = 0}$ .....(B)

Also, given $T_3 = T_4$

⇨ $k_3(x_3 - x_2 - l_3) = k_4(L - x_3 - l_4)$

now using (1), (2), and (3)

⇨ $k(x_3 - x_2 - l) = k(4l - x_3 - l)$

⇨ $kx_3 - kx_2 - kl = 4kl - kx_3 - kl$

⇨ $kx_3 + kx_3 - kx_2 = 4kl$

⇨ $2kx_3 - kx_2 = 4kl$

⇨ $k(2x_3 - x_2) = 4kl$

⇨ $\mathbf{0x_1 - x_2 + 2x_3 = 4l}$ .....(C)

Equations (A), (B), and (C) can be written in the form of a Matrix as follows

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = l \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

Taking l outside to enable calculation via Matlab

Mat_A Mat_B Mat_C

Using Matlab we solve for Mat_B (refer to Appendix 2, Page 18)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = l \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

⇨ $x_1 = l$, $x_2 = 2l$, and $x_3 = 3l$

using (3)

⇨ $\mathbf{x_1 = L/4,\ x_2 = L/2,\ and\ x_3 = 3L/4}$

d) In part (c), I deduced that $T_n = k_n(x_n - x_{n-1} - l_n)$

So, $T_1 = k_1(x_1 - x_0 - l_1) = k_1(x_1 - l_1)$

$T_2 = k_2(x_2 - x_1 - l_2)$

$T_3 = k_3(x_3 - x_2 - l_3)$

$T_4 = k_4(x_4 - x_3 - l_4)$

$T_5 = k_5(x_5 - x_4 - l_5)$

$T_6 = k_6(x_6 - x_5 - l_6) = k_6(L - x_5 - l_6)$ [because $L = x_4$]

Now, using the data given, $k_1 = 10$, $k_2 = 15$, $k_3 = 9$, $k_4 = 6$, $k_5 = 12$ and $k_6 = 19$ N.m^{-1}

$L = 1.5$ m and $l_1 = l_2 = l_3 = l_4 = l_5 = l_6 = 0.13$ m

$T_1 = 10(x_1 - 0.13)$

$T_2 = 15(x_2 - x_1 - 0.13)$

$T_3 = 9(x_3 - x_2 - 0.13)$

$T_4 = 6(x_4 - x_3 - 0.13)$

$T_5 = 12(x_5 - x_4 - 0.13)$

$T_6 = 19(1.5 - x_5 - 0.13) = 19(1.37 - x_5)$

We have to assume the system to be at rest

Therefore, $T_1 = T_2$; $T_2 = T_3$; $T_3 = T_4$; $T_4 = T_5$; $T_5 = T_6$;

For $T_1 = T_2$

⇨ $10(x_1 - 0.13) = 15(x_2 - x_1 - 0.13)$

⇨ $10x_1 - 1.3 = 15x_2 - 15x_1 - 1.95$
⇨ $25x_1 - 15x_2 = -0.65$
⇨ $5x_1 - 3x_2 = -0.13$ (dividing both side by 5)
⇨ $\mathbf{5x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5 = -0.13}$ .....(1)

For $T_2 = T_3$

⇨ $15(x_2 - x_1 - 0.13) = 9(x_3 - x_2 - 0.13)$
⇨ $15x_2 - 15x_1 - 1.95 = 9x_3 - 9x_2 - 1.17$
⇨ $-15x_1 + 24x_2 - 9x_3 = 0.78$
⇨ $5x_1 - 8x_2 + 3x_3 = -0.26$ (dividing both sides by -3)
⇨ $\mathbf{5x_1 - 8x_2 + 3x_3 + 0x_4 + 0x_5 = -0.26}$ .....(2)

For $T_3 = T_4$

⇨ $9(x_3 - x_2 - 0.13) = 6(x_4 - x_3 - 0.13)$
⇨ $9x_3 - 9x_2 - 1.17 = 6x_4 - 6x_3 - 0.78$
⇨ $-9x_2 + 15x_3 - 6x_4 = 0.39$
⇨ $3x_2 - 5x_3 + 2x_4 = -0.13$ (dividing both sides by -3)
⇨ $\mathbf{0x_1 + 3x_2 - 5x_3 + 2x_4 + 0x_5 = -0.13}$ .....(3)

For $T_4 = T_5$

⇨ $6(x_4 - x_3 - 0.13) = 12(x_5 - x_4 - 0.13)$
⇨ $6x_4 - 6x_3 - 0.78 = 12x_5 - 12x_4 - 1.56$
⇨ $-6x_3 + 18x_4 - 12x_5 = -0.78$
⇨ $-x_3 + 3x_4 - 2x_5 = -0.13$ (dividing both sides by 6)
⇨ $\mathbf{0x_1 + 0x_2 - x_3 + 3x_4 - 2x_5 = -0.13}$ .....(4)

For $T_5 = T_6$

⇨ $12(x_5 - x_4 - 0.13) = 19(1.37 - x_5)$
⇨ $12x_5 - 12x_4 - 1.56 = 26.03 - 19x_5$
⇨ $-12x_4 + 31x_5 = 27.59$
⇨ $\mathbf{0x_1 + 0x_2 + 0x_3 - 12x_4 + 31x_5 = 27.59}$ .....(5)

Equations (1), (2), (3), (4), and (5) can be written in the form of a matrix as follows

$$\begin{pmatrix} 5 & -3 & 0 & 0 & 0 \\ 5 & -8 & 3 & 0 & 0 \\ 0 & 3 & -5 & 2 & 0 \\ 0 & 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & -12 & 31 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -0.13 \\ -0.26 \\ -0.13 \\ -0.13 \\ 27.59 \end{pmatrix}$$

Mat_A Mat_B Mat_C

Using Matlab we solve for Mat_B (refer to Appendix 3, Page 18)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0.2541 \\ 0.4668 \\ 0.7346 \\ 1.0713 \\ 1.3047 \end{pmatrix}$$

⇨ **$x_1 = 0.2541$, $x_2 = 0.4668$, $x_3 = 0.7346$, $x_4 = 1.0713$, and $x_5 = 1.3047$**

Model 2: Stability

a) We are given $\zeta = c/2m\omega$

According to the question ζ is dimensionless

This implies that 'c' has the same dimensions as 2mω

Now, ω = $\sqrt{k/m}$

'k' is elasticity with units Nm^{-1}

N is the unit for force thus has dimensions MLT^{-2} (as proven in 1 (a))

Thus, dimensions of k = $MLT^{-2}/L = MT^{-2}$

So, dimensions of ω = $\sqrt{MT^{-2}/M}$ = T^{-1}

Finally, dimensions of c = dimensions of 2mω = M . T^{-1}

Therefore, dimensions of c = **MT^{-1} or M/T**

b) According to equation (E. 3),

$$x(t) = C_1 \exp[(-\zeta + \sqrt{\zeta^2 - 1})\omega t] + C_2 \exp[(-\zeta - \sqrt{\zeta^2 - 1})\omega t]$$

differentiating with respect to t

$$\dot{x}(t) = \frac{dx}{dt} = (-\boldsymbol{\zeta} + \sqrt{\boldsymbol{\zeta^2 - 1}})\boldsymbol{\omega C_1 \exp}[(-\boldsymbol{\zeta} + \sqrt{\boldsymbol{\zeta^2 - 1}})\boldsymbol{\omega t}] + (-\boldsymbol{\zeta} - \sqrt{\boldsymbol{\zeta^2 - 1}})\boldsymbol{\omega C_2 \exp}[(-\boldsymbol{\zeta} - \sqrt{\boldsymbol{\zeta^2 - 1}})\boldsymbol{\omega t}]$$

[Using, chain rule ;

$d/dx(e^x) = e^x$;

$d/dx(e^{f(x)}) = e^{f(x)}.f'(x)$]

c) i) According to equation (E. 3),

$$x(t) = C_1 \exp[(-\zeta + \sqrt{\zeta^2 - 1})\omega t] + C_2 \exp[(-\zeta - \sqrt{\zeta^2 - 1})\omega t]$$

We are given $x_0 = x(t=0) = C_1 \exp(0) + C_2 \exp(0)$

⇨ $x_0 = C_1 + C_2$

⇨ $C_1 = x_0 - C_2$ ......(1)

⇨ $C_2 = x_0 - C_1$ ......(2)

From (b) we have $\dot{x}(t) = (-\zeta + \sqrt{\zeta^2 - 1})\omega C_1 \exp[(-\zeta + \sqrt{\zeta^2 - 1})\omega t] + (-\zeta - \sqrt{\zeta^2 - 1})\omega C_2 \exp[(-\zeta - \sqrt{\zeta^2 - 1})\omega t]$

We are also given $\dot{x}(t = 0) = \dot{x}_0$

⇨ $\dot{x}_0 = (-\zeta + \sqrt{\zeta^2 - 1})\omega C_1 \exp(0) + (-\zeta - \sqrt{\zeta^2 - 1})\omega C_2 \exp(0)$

⇨ $\dot{x}_0 = (-\zeta + \sqrt{\zeta^2 - 1})\omega C_1 + (-\zeta - \sqrt{\zeta^2 - 1})\omega C_2$ ......(3)

For equation of C_1, put value from equation (2) in equation (3)

⇨ $\dot{x}_0 = (-\zeta + \sqrt{\zeta^2 - 1})\omega C_1 + (-\zeta - \sqrt{\zeta^2 - 1})\omega(x_0 - C_1)$

⇨ $\dot{x}_0 = -\zeta\omega C_1 + (\sqrt{\zeta^2 - 1})\omega C_1 - \zeta\omega x_0 - (\sqrt{\zeta^2 - 1})\omega x_0 + \zeta\omega C_1 + (\sqrt{\zeta^2 - 1})\omega C_1$

⇨ $\dot{x}_0 = 2(\sqrt{\zeta^2 - 1})\omega C_1 - \zeta\omega x_0 - (\sqrt{\zeta^2 - 1})\omega x_0$

⇨ $2(\sqrt{\zeta^2 - 1})\omega C_1 = \dot{x}_0 + \zeta\omega x_0 + (\sqrt{\zeta^2 - 1})\omega x_0$

⇨ $C_1 = \frac{\dot{x}_0}{2(\sqrt{\zeta^2-1})\omega} + \frac{\zeta\omega x_0}{2(\sqrt{\zeta^2-1})\omega} + \frac{(\sqrt{\zeta^2-1})\omega x_0}{2(\sqrt{\zeta^2-1})\omega}$

⇨ $$\boxed{\mathbf{C_1 = \frac{\dot{x}_0}{2\left(\sqrt{\zeta^2-1}\right)\omega} + \frac{\zeta x_0}{2\left(\sqrt{\zeta^2-1}\right)} + \frac{x_0}{2}}}$$

For equation of C_2, put value from equation (1) in equation (3)

⇨ $\dot{x}_0 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega(x_0 - C_2) + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega C_2$

⇨ $\dot{x}_0 = -\zeta\omega x_0 + \left(\sqrt{\zeta^2 - 1}\right)\omega x_0 + \cancel{\zeta\omega C_2} - \left(\sqrt{\zeta^2 - 1}\right)\omega C_2 - \cancel{\zeta\omega C_2} - \left(\sqrt{\zeta^2 - 1}\right)\omega C_2$

⇨ $\dot{x}_0 = -2\left(\sqrt{\zeta^2 - 1}\right)\omega C_2 - \zeta\omega x_0 + \left(\sqrt{\zeta^2 - 1}\right)\omega x_0$

⇨ $2\left(\sqrt{\zeta^2 - 1}\right)\omega C_2 = -\dot{x}_0 - \zeta\omega x_0 + \left(\sqrt{\zeta^2 - 1}\right)\omega x_0$

⇨ $C_2 = \frac{-\dot{x}_0}{2\left(\sqrt{\zeta^2-1}\right)\omega} - \frac{\zeta\cancel{\omega} x_0}{2\left(\sqrt{\zeta^2-1}\right)\cancel{\omega}} + \frac{\cancel{\left(\sqrt{\zeta^2-1}\right)\omega} x_0}{2\cancel{\left(\sqrt{\zeta^2-1}\right)\omega}}$

⇨ $$\boxed{\mathbf{C_2 = \frac{-\dot{x}_0}{2\left(\sqrt{\zeta^2-1}\right)\omega} - \frac{\zeta x_0}{2\left(\sqrt{\zeta^2-1}\right)} + \frac{x_0}{2}}}$$

ii) Condition given is $\zeta = 1$

In b) we found $\dot{x}(t) = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega C_1 \exp\left[\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega t\right] + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega C_2 \exp\left[\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega t\right]$

Applying the condition we get

$\dot{x}(t) = \left(-1 + \sqrt{1^2 - 1}\right)\omega C_1 \exp\left[\left(-1 + \sqrt{1^2 - 1}\right)\omega t\right] + \left(-1 - \sqrt{1^2 - 1}\right)\omega C_2 \exp\left[\left(-1 - \sqrt{1^2 - 1}\right)\omega t\right]$

⇨ $\dot{x}(t) = (-1)\omega C_1 \exp[(-1)\omega t] + (-1)\omega C_2 \exp[(-1)\omega t]$

⇨ $\dot{x}(t) = -\omega C_1 \exp[-\omega t] - \omega C_2 \exp[-\omega t]$

Given $\dot{x}(t = 0) = \dot{x}_0$

⇨ $\dot{x}_0 = -\omega C_1 \exp[0] - \omega C_2 \exp[0]$

⇨ $\dot{x}_0 = -\omega C_1 - \omega C_2$ .....(1)

And from c) i) we have $x_0 = C_1 + C_2$ .....(2)

Equations (1) and (2) can be written in the form of a matrix as follows

$$\begin{pmatrix} -\omega & -\omega \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_0 \\ x_0 \end{pmatrix}$$

Mat_A Mat_B Mat_C

We solve the Matrices for C_1 and C_2 using Matlab (refer to Appendix 4, Page 19)

We get that **C_1 and C_2 cannot have finite values**

Alternatively, the determinant of Mat_A = $[-\omega + \omega]$ = 0

Thus, inverse of Mat_A does not exist

So, **C_1 and C_2 have infinite solutions**

d) According to E.4

$$x(t) = C_1 \exp[(-\zeta + i\sqrt{1-\zeta^2})\omega t] + C_2 \exp[(-\zeta - i\sqrt{1-\zeta^2})\,\omega t]$$

⇨ $x(t) = C_1 e^{-\zeta\omega t} e^{i(\sqrt{1-\zeta^2}\omega t)} + C_2 e^{-\zeta\omega t} e^{-i\left(\sqrt{1-\zeta^2}\omega t\right)}$

Now we use Euler's Formula: $\mathbf{e^{i\theta} = \cos\theta + i\sin\theta}$

⇨ $x(t) = C_1 e^{-\zeta\omega t}\left[\cos\left(\sqrt{1-\zeta^2}\omega t\right) + i\sin\left(\sqrt{1-\zeta^2}\omega t\right)\right] +$

$C_2 e^{-\zeta\omega t}\left[\cos\left(-\sqrt{1-\zeta^2}\omega t\right) + i\sin\left(-\sqrt{1-\zeta^2}\omega t\right)\right]$

We know that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

⇨ $x(t) = C_1 e^{-\zeta\omega t}\left[\cos\left(\sqrt{1-\zeta^2}\omega t\right) + i\sin\left(\sqrt{1-\zeta^2}\omega t\right)\right] +$

$C_2 e^{-\zeta\omega t}\left[\cos\left(\sqrt{1-\zeta^2}\omega t\right) - i\sin\left(-\sqrt{1-\zeta^2}\omega t\right)\right]$

⇨ $x(t) = C_1 e^{-\zeta\omega t}\cos\left(\sqrt{1-\zeta^2}\omega t\right) + C_1 e^{-\zeta\omega t} i\sin\left(\sqrt{1-\zeta^2}\omega t\right) +$

$C_2 e^{-\zeta\omega t}\cos\left(\sqrt{1-\zeta^2}\omega t\right) - C_2 e^{-\zeta\omega t} i\sin\left(-\sqrt{1-\zeta^2}\omega t\right)$

⇨ $x(t) = \cos\left(\sqrt{1-\zeta^2}\omega t\right)[\, C_1 e^{-\zeta\omega t} + C_2 e^{-\zeta\omega t}]$

$+ i\sin\left(\sqrt{1-\zeta^2}\omega t\right)[C_1 e^{-\zeta\omega t} - C_2 e^{-\zeta\omega t}]$

⇨ $x(t) = e^{-\zeta\omega t}.\cos\left(\sqrt{1-\zeta^2}\omega t\right)[\, C_1 + C_2] + e^{-\zeta\omega t}.\, i\sin\left(\sqrt{1-\zeta^2}\omega t\right)$

$[C_1 - C_2]$

Now Replacing with $C_1' = C_1 + C_2$ and $C_2' = i(C_1 - C_2)$

⇨ $x(t) = e^{-\zeta\omega t}.\cos\left(\sqrt{1-\zeta^2}\omega t\right)\underline{[\,\mathbf{C_1} + \mathbf{C_2}]} + e^{-\zeta\omega t}.\underline{\mathbf{i}}\sin\left(\sqrt{1-\zeta^2}\omega t\right)$

$\underline{[\mathbf{C_1} - \mathbf{C_2}]}$

⇨ $$\boxed{\mathbf{x(t) = \exp(-\zeta\omega t)\,\{C_1' \cos\left[\left(\sqrt{1-\zeta^2}\right)\omega t\right] + C_2' \sin\left[\left(\sqrt{1-\zeta^2}\right)\omega t\right]\}}}$$

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e) $x(t) = \exp(-\zeta\omega t)\,\{C_1' \cos\left[\left(\sqrt{1-\zeta^2}\right)\omega t\right] + C_2' \sin\left[\left(\sqrt{1-\zeta^2}\right)\omega t\right]\}$

⇨ $x(t=0) = \exp(-\zeta\omega(0))\,\{C_1' \cos\left[\left(\sqrt{1-\zeta^2}\right)\omega(0)\right] +$

$C_2' \sin\left[\left(\sqrt{1-\zeta^2}\right)\omega(0)\right]\}$

⇨ $x_0 = C_1' \cos(0) + C_2' \sin(0)$

⇨ $x_0 = C_1'$ [because cos(0) = 1 and sin(0) = 0]........(1)

Now, x(t) is of the form u(t).v(t)

(Where, u(t) = $\exp(-\zeta\omega t)$ and

v(t) = $\{C_1' \cos\left[\left(\sqrt{1-\zeta^2}\right)\omega t\right]\; C_2' \sin\left[\left(\sqrt{1-\zeta^2}\right)\omega t\right]\}$)

To differentiate we will apply product rule
i.e d/dx(u.v) = u'v +uv'

Differentiating E.5 with respect to 't'

$$\dot{x}(t) = \frac{dx}{dt} = -\zeta\omega.\exp(-\zeta\omega t)\{C_1'\cos[(\sqrt{1-\zeta^2})\omega t] + C_2'\sin[(\sqrt{1-\zeta^2})\omega t]\} + \exp(-\zeta\omega t).(\sqrt{1-\zeta^2})\omega\{C_2'\cos[(\sqrt{1-\zeta^2})\omega t] - C_1'\sin[(\sqrt{1-\zeta^2})\omega t]\}$$

Using, $d/dx(e^{f(x)}) = e^{f(x)}.f'(x)$ chain rule
$d/dx(\sin x) = \cos x$
$d/dx(\cos x) = -\sin x$
$d/dx(e^x) = e^x$

⇨ $\dot{x}(t=0) = \dot{x}_0 = -\zeta\omega.\exp(0)\{C_1'\cos[0] + C_2'\sin[0]\} + \exp(0).(\sqrt{1-\zeta^2})\omega\{C_2'\cos[0] - C_1'\sin[0]\}$

⇨ $\dot{x}_0 = -\zeta\omega\{C_1'\} + (\sqrt{1-\zeta^2})\omega\{C_2'\}$

Putting value of $C_1' = x_0$ from (1)

⇨ $\dot{x}_0 = -\zeta\omega x_0 + (\sqrt{1-\zeta^2})\omega\{C_2'\}$

⇨ $(\sqrt{1-\zeta^2})\omega\{C_2'\} = \dot{x}_0 + \zeta\omega x_0$

⇨ $$\mathbf{C_2' = \frac{\dot{x}_0 + \zeta\omega x_0}{(\sqrt{1-\zeta^2})\omega}} \quad .....(2)$$

And also from (1)

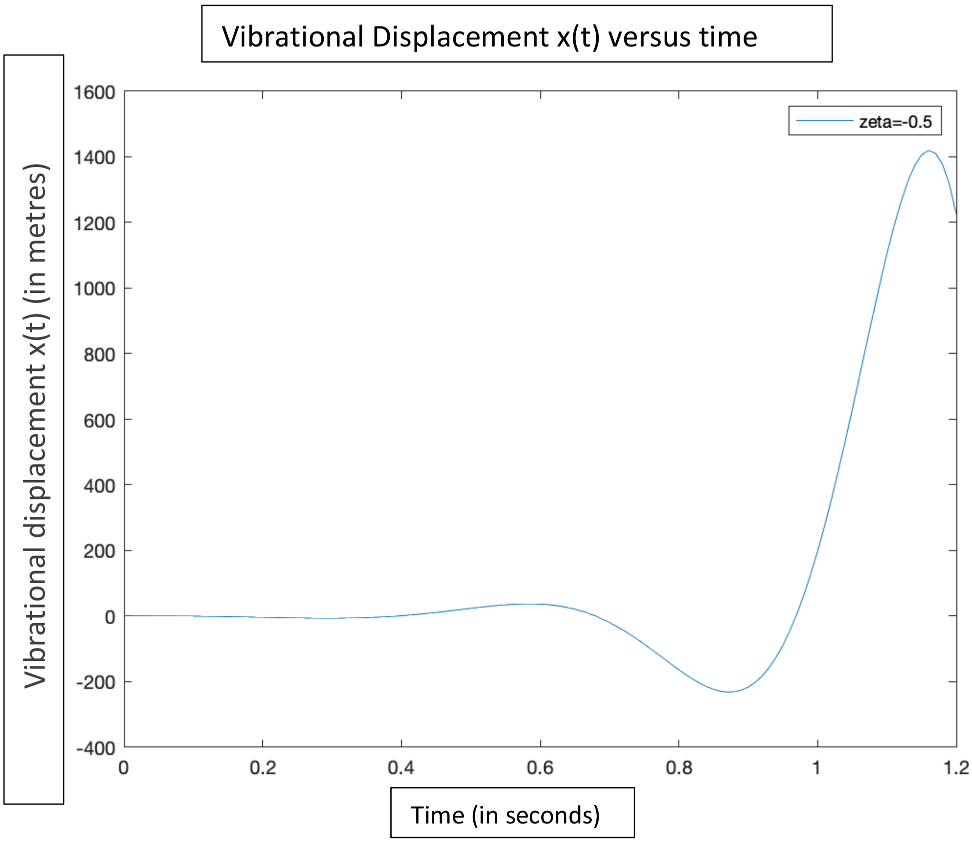
⇨ $\mathbf{C_1' = x_0}$

In order to obtain a fully explicit form of E.5 we must input values of C_1' and C_2' from (1) and (2) into E.5

$$x(t) = \exp(-\zeta\omega t)\{C_1'\cos[(\sqrt{1-\zeta^2})\omega t] + C_2'\sin[(\sqrt{1-\zeta^2})\omega t]\}$$

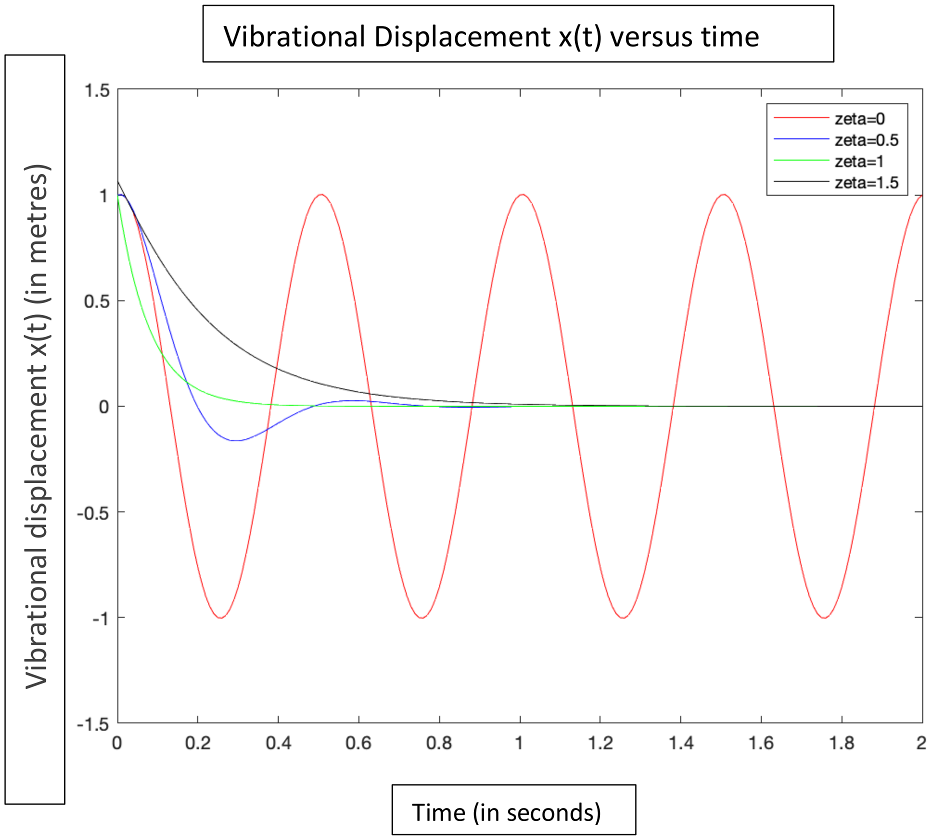
⇨ $$x(t) = \exp(-\zeta\omega t)\{x_0\cos[(\sqrt{1-\zeta^2})\omega t] + \frac{\dot{x}_0 + \zeta\omega x_0}{(\sqrt{1-\zeta^2})\omega}\sin[(\sqrt{1-\zeta^2})\omega t]\}$$

f) (For the Matlab Code and process of plotting the graph refer to Appendix 5, Page 20)



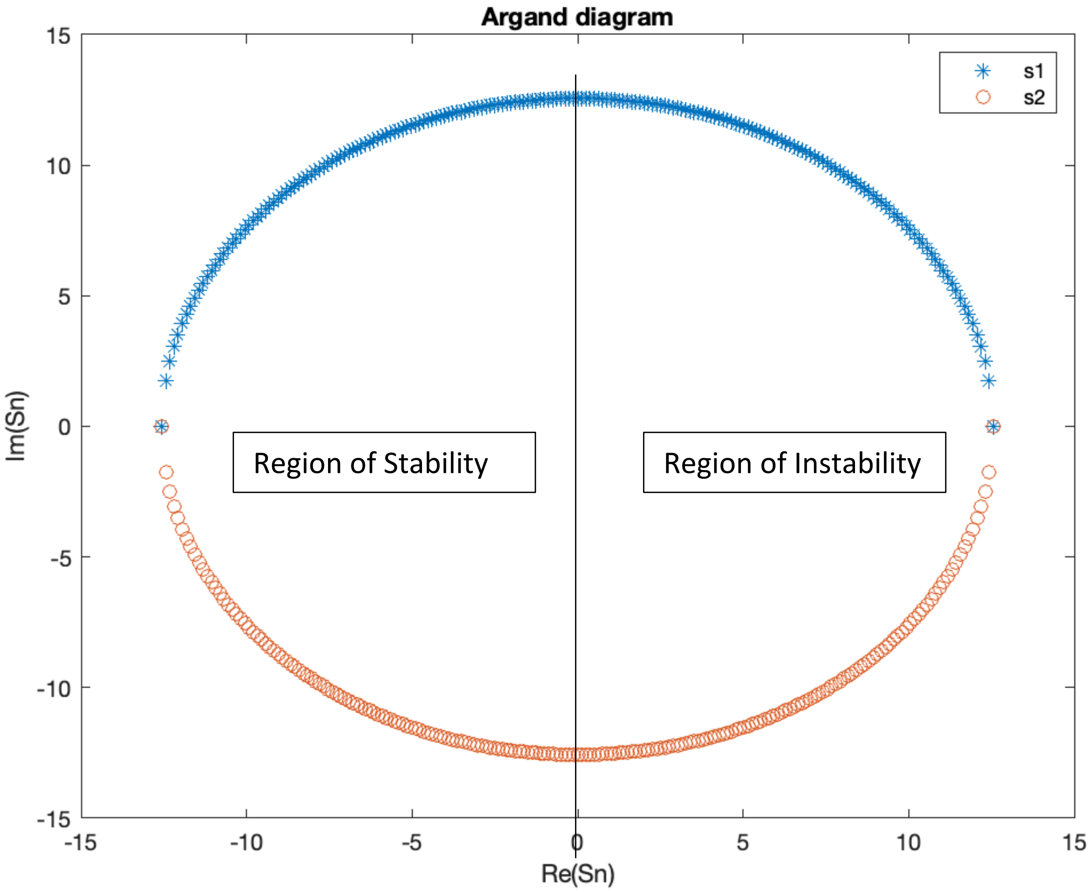
The Graph above represents Vibrational Displacement as a function of time. According to this graph, we see how the vibrational displacement varies with time for a fixed value of **ζ < 0**, i.e ζ = -0.5. The vibrational displacement stays approximately constant at 0 till it momentarily increase at t=0.6 seconds and then moves to negative displacement till about t=0.9 seconds and then rapidly increases reaching its peak a little before t = 1.2 seconds and then continues this cycle in sinusoidal pattern. We can hence observe that the **amplitude of natural oscillatory displacement increases continuously** with time hence the system is considered to be **dynamically unstable**.

(See Next Page For The Other Graph)



This Graph shows how the Vibrational Displacement varies with time for varying values of ζ=0 and ζ>0. In the graph for **ζ=0** we observe a sinusoidal graph with a **constant amplitude of 1 m**. This implies that the **system is dynamically stable** as the amplitude of natural oscillations remains steady with varying time. The graph of values of ζ >0 i.e ζ=0.5, ζ=1 and ζ=1.5 the graph starts with displacement of 1 m at t=0 then declines to 0 m between t=0.2 seconds and t=0.4 seconds after which it remains constant at zero displacement. Thus, the graphs of **ζ>0** show a **decrease in the amplitude of oscillatory displacement with time till it stabilizes at 0** and hence the system is considered to be **dynamically stable**.

g) (For the Matlab Code and process of plotting the graph refer to Appendix 6, Page 21)



From the code for the graph plotted on Matlab, it was observed that the very first values of s1 and s2 on the graph are plotted at (12.5664,0) and the last values of s1 and s2 on the graph are plotted at (-12.5664,0). This fact gives us assistance with the fact that we must read the graph from right to left i.e from positive to negative. The values of ζ range from -1 to +1. If we were to graph the line x=0 it would divide the diagram into two parts, the part with positive ζ and the part with negative ζ . Also, we concluded in part f) that when $\zeta<0$ the system is dynamically unstable and that when $\zeta >=0$ it is dynamically stable. Keeping this fact in mind we can conclude that the region on the right is dynamically unstable and the region on the left is dynamically stable. We were informed that the Tacoma Bridge was self excited and grew unbounded which leads us to conclude that it will be located in a region of instability and hence it can be said that it is located on the right side of the graph.

Model 3: Flow

a) Given, $h(t) = \left[h_0^{\frac{5}{2}} - \frac{5a^{\Phi}\sqrt{2g}}{2\tan^2(\frac{\pi}{2} - \theta)}t\right]^{\frac{2}{5}}$

$h_0^{\frac{5}{2}}$ and $\frac{5a^{\Phi}\sqrt{2g}}{2\tan^2(\frac{\pi}{2} - \theta)}t$ must have the same dimensions because you can only add or subtract quantities with the same dimensions.

Also, In this equation '$2\tan^2(\frac{\pi}{2} - \theta)$' is dimensionless because it's a trigonometric function.

Now, a has dimensions 'L' and g is acceleration [speed(LT^{-1}) per unit time(T)] so has dimensions 'LT^{-2}' and t has dimensions 'T'

Thus, dimensions of $\frac{5a^{\Phi}\sqrt{2g}}{2\tan^2(\frac{\pi}{2} - \theta)}t = L^{\Phi} . (LT^{-2})^{1/2} . T = L^{\Phi+1/2}$

Now, h_0 is a height thus its dimension is 'L' so the dimensions of $h_0^{\frac{5}{2}}$ are '$L^{5/2}$'

Dimensions of $\frac{5a^{\Phi}\sqrt{2g}}{2\tan^2(\frac{\pi}{2} - \theta)}t$ = Dimensions of $h_0^{\frac{5}{2}}$

⇨ $L^{\varphi+1/2} = L^{5/2}$

Comparing powers

⇨ $\Phi + 1/2 = 5/2$

⇨ **$\Phi = 2$**

b) Volume of a cone = $\frac{\pi r^2 h}{3}$

Given, $r(t)/h(t) \approx \tan(\pi/2 - \theta)$

⇨ $r(t) \approx h(t).\tan(\pi/2 - \theta)$ .....(1)

Now, $V(t) = \frac{\pi r(t)^2 h(t)}{3}$ ......(2)

Putting value of r(t) from equation (1) in equation (2)

$V(t) \approx \frac{\pi(h(t).\tan\left(\frac{\pi}{2} - \theta\right))^2 h(t)}{3}$

⇨ $$\mathbf{V(t) \approx \frac{\pi.(h(t))^3(\tan\left(\frac{\pi}{2} - \theta\right))^2}{3}}$$

c) We have $h(t) = \left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2} - \theta)}t\right]^{\frac{2}{5}}$

This expression shows us the height h of the water in the container as a function of the time t

Let τ be the time the tank takes to be empty

Thus, when t = τ

h(τ) = 0

⇨ $\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2} - \theta)}\tau\right]^{\frac{2}{5}} = 0$

Taking the 5/2th power of both sides

⇨ $h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}\tau = 0$

⇨ $h_0^{\frac{5}{2}} = \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}\tau$

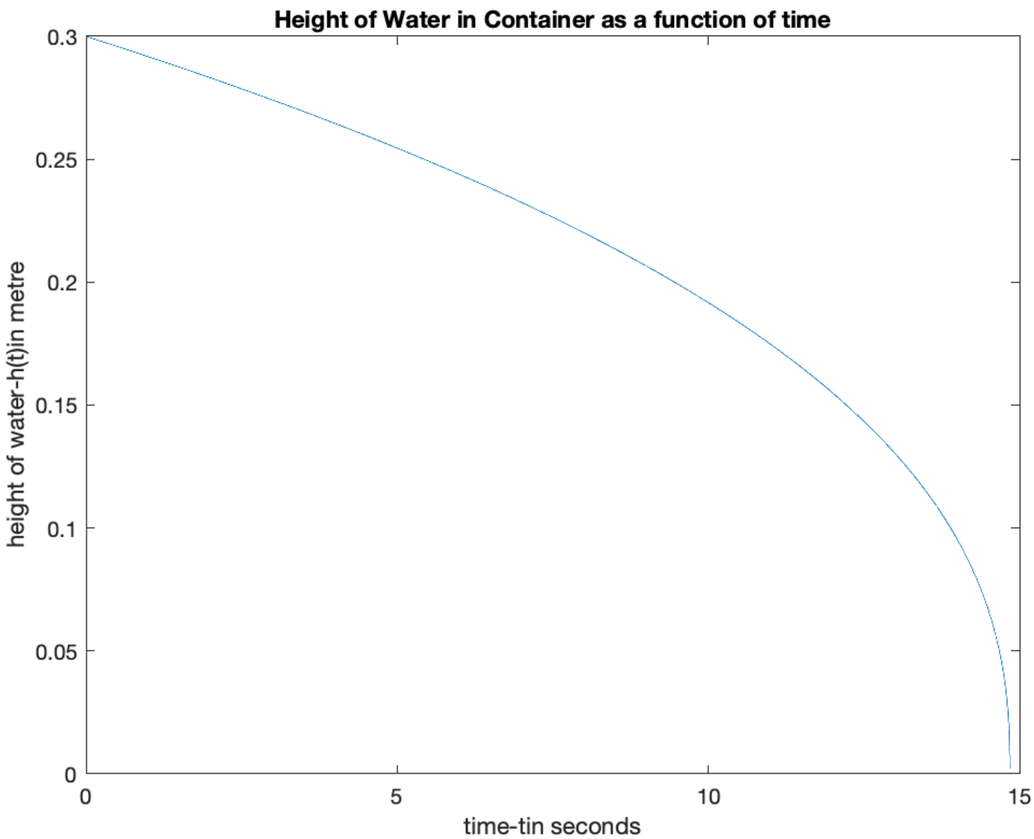
⇨ $\boxed{\boldsymbol{\tau = h_0^{\frac{5}{2}}\frac{2\tan^2(\frac{\pi}{2}-\theta)}{5a^2\sqrt{2g}}}}$

We are given h_0 = 30 cm = 0.3 m and a = 1 cm = 0.01 m
'g' is acceleration due to gravity thus, g = 9.8 m/s
Let us assume θ to be an arbitrary angle say $\theta = \pi/3$

⇨ $\tau = (0.3)^{\frac{5}{2}}\frac{2\tan^2(\frac{\pi}{2}-\frac{\pi}{3})}{5(0.01)^2\sqrt{2(9.8)}}$

⇨ $\boxed{\tau = 14.8461 \text{ s}}$

(For the code of the graph refer to appendix 7, Page 22)



The graph shows linear declination in height of water in container with time.
To check the validity we can observe that the graph ends at $\tau = 14.8461$ s which is the same as the value of τ calculated so it stands valid.

d) We have $h(t) = \left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2} - \theta)} t\right]^{\frac{2}{5}}$ .....(1)

And $\tau = h_0^{\frac{5}{2}} \frac{2\tan^2(\frac{\pi}{2} - \theta)}{5a^2\sqrt{2g}}$

⇨ $5a^2\sqrt{2g} = \frac{2\tan^2(\frac{\pi}{2} - \theta)}{\tau} h_0^{\frac{5}{2}}$ .....(2)

Replacing the value of $5a^2\sqrt{2g}$ in (1) from (2)

$$h(t) = \left[h_0^{\frac{5}{2}} - \frac{\frac{\cancel{2\tan^2(\frac{\pi}{2} - \theta)}}{\tau} h_0^{\frac{5}{2}}}{\cancel{2\tan^2(\frac{\pi}{2} - \theta)}} t\right]^{\frac{2}{5}}$$

⇨ $h(t) = \left[h_0^{\frac{5}{2}} - \frac{h_0^{\frac{5}{2}}}{\tau} t\right]^{\frac{2}{5}}$

Taking $h_0^{\frac{5}{2}}$ common

⇨ $\boxed{h(t) = h_0 \left[1 - \frac{t}{\tau}\right]^{\frac{2}{5}}}$ .....(3)

Now, differentiating (3) with respect to t

$$\boxed{\dot{h}(t) = \frac{dh}{dt} = -\frac{2}{5} h_0 \left[1 - \frac{t}{\tau}\right]^{-\frac{3}{5}}}$$

[Using, chain rule ;

$d/dx(x^n) = n.x^{n-1}$

$d/dx(e^{f(x)}) = e^{f(x)}.f'(x)$

$d/dx(a.f(x)) = a. f'(x)$ where a is a constant]

e) $V(t) \approx \frac{\pi(h(t))^3 . \tan^2(\frac{\pi}{2} - \theta)}{3}$

For the sake of simplicity lets have

⇨ $V(t) = \frac{\pi}{3}(h(t))^3 . \tan^2\left(\frac{\pi}{2} - \theta\right)$

$$\Rightarrow \quad V(t) = \frac{\pi}{3}\left(\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}t\right]^{\frac{2}{5}}\right)^3 .\tan^2\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \quad V(t) = \frac{\pi}{3}\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}t\right]^{\frac{6}{5}} .\tan^2\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \quad V(t) = \frac{\pi}{3}\tan^2\left(\frac{\pi}{2} - \theta\right)\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}t\right]^{\frac{6}{5}} .$$

Constant Terms

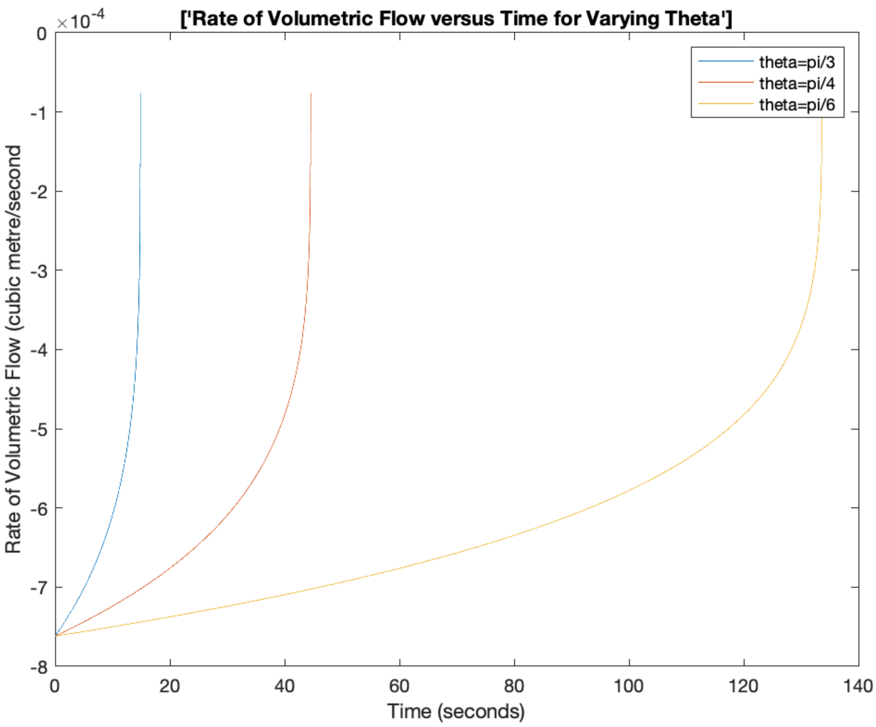
Differentiating with respect to t
Using chain rule and the fact that d/dx (x^n) = nx^{n-1}

$$\Rightarrow \quad \dot{V}(t) = \frac{dv}{dt} = \frac{\pi}{3}\tan^2\left(\frac{\pi}{2} - \theta\right)\frac{6}{5}\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}t\right]^{\frac{1}{5}}\left(0 - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}\right)$$

$$\Rightarrow \quad \dot{V}(t) = \frac{\pi}{3}\tan^2\left(\frac{\pi}{2} - \theta\right)\frac{6}{5}\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}t\right]^{\frac{1}{5}}\left(- \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}\right)$$

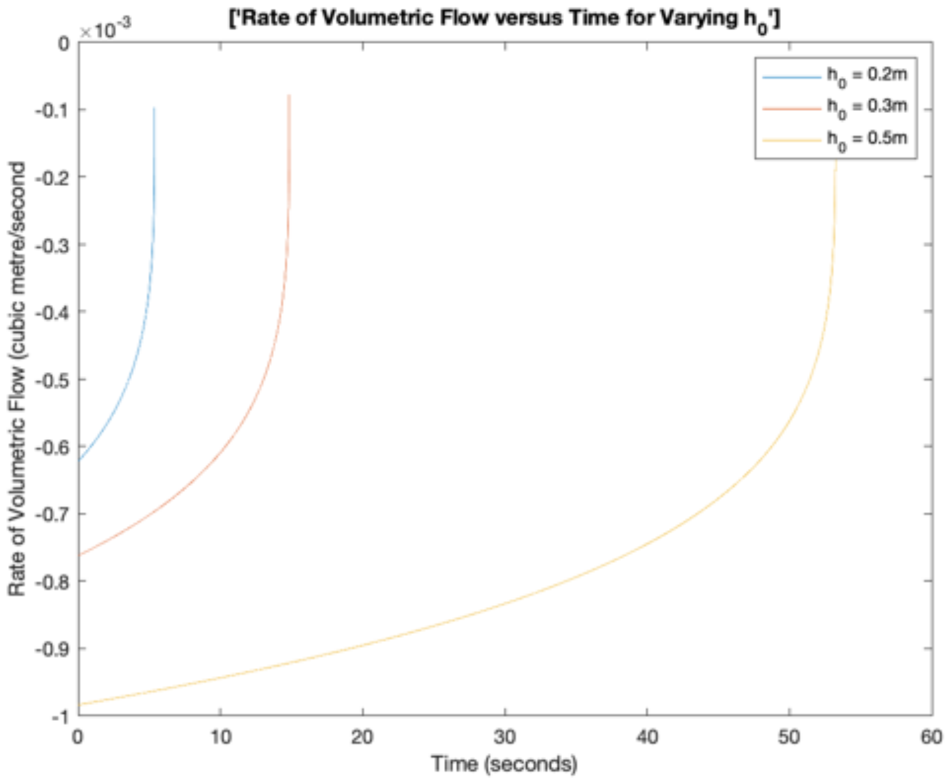
$$\Rightarrow \quad \dot{V}(t) = -\frac{\pi}{\cancel{3}}\cancel{\tan^2\left(\frac{\pi}{2} - \theta\right)}.\frac{\cancel{6}}{\cancel{5}}\left(\frac{\cancel{5}a^2\sqrt{2g}}{\cancel{2\tan^2(\frac{\pi}{2}-\theta)}}\right)\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}t\right]^{\frac{1}{5}}$$

$$\Rightarrow \quad \boxed{\dot{V}(t) = -\pi a^2\sqrt{2g}\left[h_0^{\frac{5}{2}} - \frac{5a^2\sqrt{2g}}{2\tan^2(\frac{\pi}{2}-\theta)}t\right]^{\frac{1}{5}}}$$

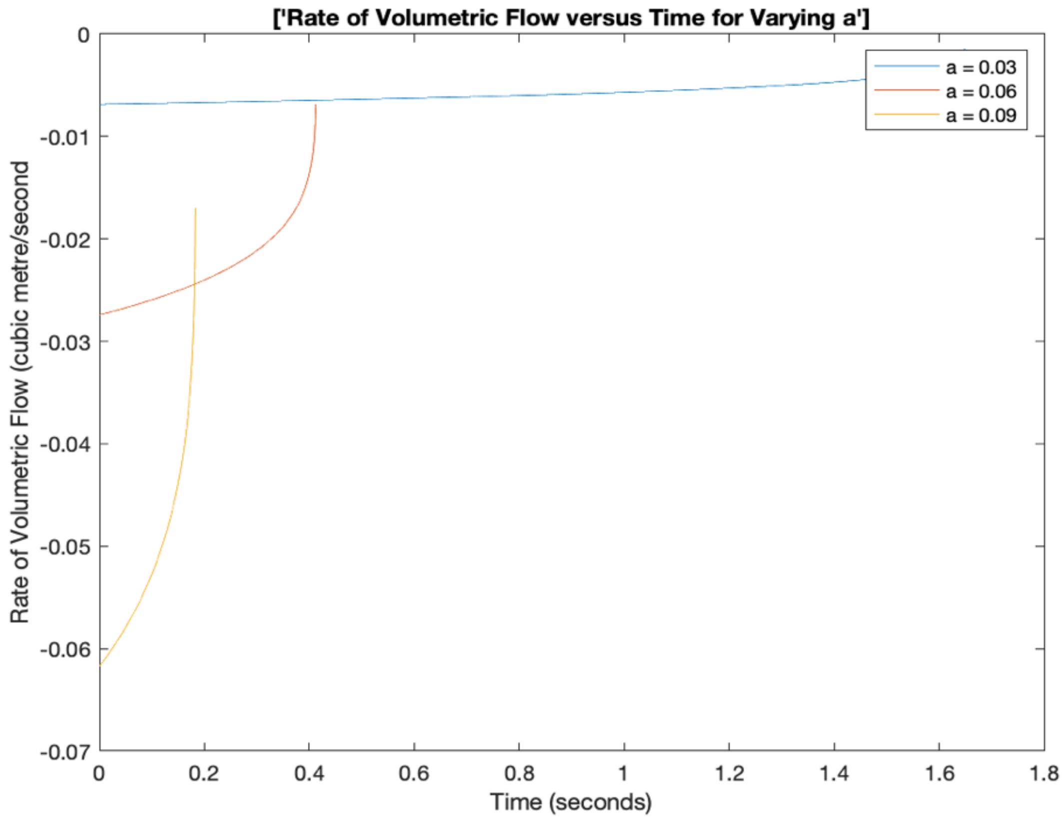


Please turn over

(Refer to Appendix 8 i) page 23)
θ : From this graph we observe that as the angle θ of the container decreases, rate of volumetric flow decreases with respect to time. This means that as the as the angle θ decreases, the outflow decreases, and it takes longer to empty the container. Eg, for theta=pi/3 container empties in approx. 18 secs and theta=pi/6 takes about 130 secs.



(Refer to Appendix 8 ii) page 25)
h_0: From this graph we observe that as the height h_0 increases, rate of volumetric flow decreases with respect to time. This means that as the height h_0 increases, the outflow decreases, and it takes longer to empty the container. Eg, for h_0=0.2m container takes about 5 seconds to empty itself and for h_0=0.5m container takes about 55 seconds.



(Refer to Appendix 8 iii) page 27)

a: From this graph we observe that as outlet ‘a’ increases, rate of volumetric flow increases with respect to time. This means that as outlet ‘a’ increases, the outflow increases and the container is emptied faster. Eg, when a=0.03m it takes about 1.7 seconds for the container to empty itself whereas when a=0.09m it takes about 0.2 seconds for it to empty itself.

APPENDIX

1. First we assign the values to Matrix A and Matrix C

```
Mat_A = [2,-1;-1,2]
```

```
Mat_A = 2×2
      2    -1
     -1     2
```

```
Mat_C = [0;3]
```

```
Mat_C = 2×1
      0
      3
```

In order to evaluate the Matrix b we need to multiply the inverse of Matrix A with Matrix C

```
Mat_b = inv(Mat_A)*(Mat_C)
```

```
Mat_b = 2×1
      1
      2
```

2. First we assign the values to Matrix A and Matrix C

```
Mat_A = [2,-1,0;-1,2,-1;0,-1,2]
```

```
Mat_A = 3×3
      2    -1     0
     -1     2    -1
      0    -1     2
```

```
Mat_C = [0;0;4]
```

```
Mat_C = 3×1
      0
      0
      4
```

In order to evaluate the Matrix b we need to multiply the inverse of Matrix A with Matrix C

```
Mat_b = inv(Mat_A)*(Mat_C)
```

```
Mat_b = 3×1
    1.0000
    2.0000
    3.0000
```

3. First we assign the values to Matrix A and Matrix C

```
Mat_A = [5,-3,0,0,0;5,-8,3,0,0;0,3,-5,2,0;0,0,-1,3,-2;0,0,0,-12,31]
```

```
Mat_A = 5×5
      5    -3     0     0     0
```

5	-8	3	0	0
0	3	-5	2	0
0	0	-1	3	-2
0	0	0	-12	31

```
Mat_C = [-0.13;-0.26;-0.13;-0.13;27.59]
```

```
Mat_C = 5×1
   -0.1300
   -0.2600
   -0.1300
   -0.1300
   27.5900
```

In order to evaluate the Matrix b we need to multiply the inverse of Matrix A with Matrix C

```
Mat_B = inv(Mat_A)*(Mat_C)
```

```
Mat_B = 5×1
    0.2541
    0.4668
    0.7346
    1.0713
    1.3047
```

4. First we establish the variables and then we assign the values to Matrix A and Matrix C

```
syms omega
syms C1
syms C2
syms x_0
syms x_0dot
Mat_A = [-omega, -omega ; 1,1]
```

Mat_A =

$$\begin{pmatrix} -\omega & -\omega \\ 1 & 1 \end{pmatrix}$$

```
Mat_C = [x_0dot ; x_0]
```

Mat_C =

$$\begin{pmatrix} x_{0\mathrm{dot}} \\ x_0 \end{pmatrix}$$

In order to evaluate the Matrix b we need to multiply the inverse of Matrix A with Matrix C

```
Mat_b = inv(Mat_A)*Mat_C
```

Mat_b =

$$\begin{pmatrix} x_0\,\infty + x_{0\mathrm{dot}}\,\infty \\ x_0\,\infty + x_{0\mathrm{dot}}\,\infty \end{pmatrix}$$

5.
We are going to plot the vibrational displacement ‘x(t)’ as a function of time ‘t’ for different values of zeta in order to determine the effect of zeta on the stability of the system.
We are given the values of omega x_0dot and x_0 which are 4*pi,1 and 1 respectively
First we consider the value of zeta to be -0.5

```
t=0:0.01:1.2;
```

```
zeta=-0.5;
omega=4*pi;
x_0=1;
x_0dot=1;
```

The vibrational displacement ‘x(t)’ for zeta<1 is given by:

```
xt_zeta1=exp(-zeta.*omega.*t).*(x_0.*cos(sqrt(1-
zeta.^2).*omega.*t)+(x_0dot+zeta.*omega.*x_0)/(sqrt(1-
zeta.^2).*omega).*sin(sqrt(1-zeta.^2).*omega.*t));
```

Now, we plot the vibrational displacement in terms of time.
We choose to plot the first curve with value of zeta = -0.5 on a different graph for better visualisation.

```
plot(t,xt_zeta1)
legend('zeta=-0.5')
```

Similarly, we now repeat the process with varying values of zeta
Plot the curves on the same graph to compare the effect of the change of zeta on the vibrational displacement and thus the stability of the system.

```
t=0:0.01:2;
omega=4*pi;
x_0=1;
x_0dot=1;

zeta=0;
xt_zeta2=exp(-zeta.*omega.*t).*(x_0.*cos(sqrt(1-
zeta.^2).*omega.*t)+(x_0dot+zeta.*omega.*x_0)/(sqrt(1-
zeta.^2).*omega).*sin(sqrt(1-zeta.^2).*omega.*t));
plot(t,xt_zeta2,'r')
hold on
```

```
zeta=0.5;
```

```
 xt_zeta3=exp(-zeta.*omega.*t).*(x_0.*cos(sqrt(1-
zeta.^2).*omega.*t)+(x_0dot+zeta.*omega.*x_0)/(sqrt(1-
zeta.^2).*omega).*sin(sqrt(1-zeta.^2).*omega.*t));
 plot(t,xt_zeta3,'b')
 hold on
```

For zeta=1, we have deduced that x(t) is written as x(t)=x_0*exp(-omega*t)

```
zeta=1;
xt_zeta4=x_0*exp(-omega*t)
```

```
xt_zeta4 = 1×201
    1.0000    0.8819    0.7778    0.6859    0.6049    0.5335    0.4705 ⋯
```

```
plot(t,xt_zeta4,'g')
hold on
```

For zeta > 1 we have to use the expression of x(t) as it is in (E.3) since we already defined C1 and C2 in terms of x_0 and x0_dot:

```
 zeta=1.5;
 xt_zeta5=((x_0*omega*(sqrt(zeta^2-1)+zeta))+x_0dot)/(2*omega*sqrt(zeta^2-
1))*exp((-zeta+sqrt(zeta^2-1))*omega*t)+(x_0*omega*(-zeta+sqrt(zeta^2-1)-
x0_dot))/(2*omega*sqrt(zeta^2-1))*exp((-zeta-sqrt(zeta^2-1)*omega*t));
 pl
```

6.

We need to plot an argand diagram to show the effect of zeta on the position of s1 and s2 in the complex plane

Now, we define the value of zeta

```
zeta = -1:0.01:1
```

```
zeta = 1×201
   -1.0000   -0.9900   -0.9800   -0.9700   -0.9600   -0.9500   -0.9400 ⋯
```

We also define omega

```
omega = 4*pi
```

```
omega = 12.5664
```

The expression of the conjugate frequencies of damped variations s1 and s2 are defined

```
s1 = (-zeta+i.*sqrt(1-zeta.^2))*omega
```

```
s1 = 1×201 complex
   12.5664 + 0.0000i   12.4407 + 1.7727i   12.3150 + 2.5007i   12.1894 +
3.0549i ⋯
```

```
s2 = (-zeta-i.*sqrt(1-zeta.^2))*omega
```

```
s2 = 1×201 complex
   12.5664 + 0.0000i   12.4407 - 1.7727i   12.3150 - 2.5007i   12.1894 -
3.0549i ⋯
```

In order to plot the graph we must define the real and imaginary part of s1 and s2

```
real_s1 = real(s1)
```

```
real_s1 = 1×201
   12.5664   12.4407   12.3150   12.1894   12.0637   11.9381   11.8124 ⋯
```

```
real_s2 = real(s2)
```

```
real_s2 = 1×201
   12.5664   12.4407   12.3150   12.1894   12.0637   11.9381   11.8124 ⋯
```

```
imag_s1 = imag(s1)
```

```
imag_s1 = 1×201
         0    1.7727    2.5007    3.0549    3.5186    3.9238    4.2873 ⋯
```

```
imag_s2 = imag(s2)
```

```
imag_s2 = 1×201
         0   -1.7727   -2.5007   -3.0549   -3.5186   -3.9238   -4.2873 ⋯
```

Plot on the graph

```
plot(real_s1 , imag_s1, '*')
hold on
plot(real_s2 , imag_s2, 'o')
title('Argand diagram')
xlabel('Re(Sn)')
ylabel('Im(Sn)')
legend('s1','s2')
```

7. First we assign the vaue to constants

```
a = 0.01
```

```
a = 0.0100
```

```
h_0 = 0.3
```

```
h_0 = 0.3000
```

```
theta = pi/3
```

```
theta = 1.0472
```

```
g = 9.8
```

```
g = 9.8000
```

```
tau = ((h_0)^(5/2))*((2*tan(pi/2 - theta)^2)/(5*(a)^2*sqrt(2*g)))
```

```
tau = 14.8461
```

Then we set the limits for t

```
t = 0:0.0001:tau
```

```
t = 1×148462
          0    0.0001    0.0002    0.0003    0.0004    0.0005    0.0006 ⋯
```

We then write the expression for h_t

```
h_t = (((h_0)^(5/2))-((5*a^2*sqrt(2*g))*t)/(2*(tan(pi/2 - theta)^2))).^(2/5)
```

```
h_t = 1×148462
    0.3528    0.3528    0.3528    0.3528    0.3528    0.3528    0.3528 ⋯
```

Finally we plot the values on the graph

```
plot(t, h_t)
xlabel('time-t{in seconds}')
ylabel('height of water-h(t){in metre}')
title ('Height of Water in Container as a function of time')
```

8.i) For theta
we keep all other terms constant

```
a = 0.01
```

```
a = 0.0100
```

```
h_0 = 0.3
```

```
h_0 = 0.3000
```

```
g = 9.8
```

```
g = 9.8000
```

We find V_dot for varying values of theta
First we use

```
theta = pi/3
```

```
theta = 1.0472
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 14.8461
```

```
t = 0:0.001:tau
```

```
t = 1×14847
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060  ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×14847
10^-3 ×
   -0.7618   -0.7618   -0.7618   -0.7618   -0.7618   -0.7617   -0.7617  ⋯
```

Then we plot it on the graph
We repeat the same process for different values of theta
We plot these values on the same graph to facilitate better comparison

```
plot(t,v_dot)
hold on

theta = pi/4
```

```
theta = 0.7854
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 44.5384
```

```
t = 0:0.001:tau
```

```
t = 1×44539
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060  ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×44539
10^-3 ×
   -0.7618   -0.7618   -0.7618   -0.7618   -0.7618   -0.7618   -0.7618  ⋯
```

```
plot(t,v_dot)
hold on

theta = pi/6
```

```
theta = 0.5236
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 133.6153
```

```
t = 0:0.001:tau
```

```
t = 1×133616
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060 ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×133616
10^-3 ×
   -0.7618   -0.7618   -0.7618   -0.7618   -0.7618   -0.7618   -0.7618 ⋯
```

```
plot(t,v_dot)
hold on

ylabel('Rate of Volumetric Flow (cubic metre/second')
xlabel('Time (seconds)')
legend('theta=pi/3', 'theta=pi/4','theta=pi/6')
title ['Rate of Volumetric Flow versus Time for Varying Theta']
hold off
```

ii) For h_0
We keep all the other terms constant

```
a = 0.01
```

```
a = 0.0100
```

```
theta  = pi/3
```

```
theta = 1.0472
```

```
g = 9.8
```

```
g = 9.8000
```

We find V_dot for varying values of h_0
First we use

```
h_0 = 0.2
```

```
h_0 = 0.2000
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 5.3875
```

```
t = 0:0.001:tau
```

```
t = 1×5388
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060 ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×5388
10^-3 ×
   -0.6220    -0.6220    -0.6220    -0.6219    -0.6219    -0.6219    -0.6219 ⋯
```

Then we plot it on the graph
We repeat the same process for different values of h_0
We plot these values on the same graph to facilitate better comparison

```
plot(t,v_dot)
hold on

h_0 = 0.3
```

```
h_0 = 0.3000
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 14.8461
```

```
t = 0:0.001:tau
```

```
t = 1×14847
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060 ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×14847
10^-3 ×
   -0.7618    -0.7618    -0.7618    -0.7618    -0.7618    -0.7617    -0.7617 ⋯
```

```
plot(t,v_dot)
hold on

h_0 = 0.5
```

```
h_0 = 0.5000
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 53.2397
```

```
t = 0:0.001:tau
```

```
t = 1×53240
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060 ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×53240
10^-3 ×
   -0.9835    -0.9835    -0.9835    -0.9835    -0.9835    -0.9835    -0.9835 ⋯
```

```
plot(t,v_dot)
hold on

ylabel('Rate of Volumetric Flow (cubic metre/second')
xlabel('Time (seconds)')
legend('h_0 = 0.2m', 'h_0 = 0.3m','h_0 = 0.5m')
title ['Rate of Volumetric Flow versus Time for Varying h_0']
hold off
```

iii) For a

We keep all other values constant

```
theta = pi/3
```

```
theta = 1.0472
```

```
h_0 = 0.3
```

```
h_0 = 0.3000
```

```
g = 9.8
```

```
g = 9.8000
```

We find V_dot for varying values of a

First we use

```
a = 0.03
```

```
a = 0.0300
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 1.6496
```

```
t = 0:0.001:tau
```

```
t = 1×1650
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060 ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×1650
   -0.0069   -0.0069   -0.0069   -0.0069   -0.0069   -0.0069   -0.0069 ⋯
```

Then we plot it on the graph

We repeat the same process for different values of a

We plot these values on the same graph to facilitate better comparison

```
plot(t,v_dot)
hold on
```

```
a = 0.06
```

```
a = 0.0600
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 0.4124
```

```
t = 0:0.001:tau
```

```
t = 1×413
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060 ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×413
   -0.0274   -0.0274   -0.0274   -0.0274   -0.0274   -0.0274   -0.0273 ⋯
```

```
plot(t,v_dot)
hold on

a = 0.09
```

```
a = 0.0900
```

```
tau = h_0^(5/2)*(2*(tan(pi/2-theta))^2/(5*a^2*sqrt(2*g)))
```

```
tau = 0.1833
```

```
t = 0:0.001:tau
```

```
t = 1×184
         0    0.0010    0.0020    0.0030    0.0040    0.0050    0.0060 ⋯
```

```
v_dot = -pi.*a^2.*sqrt(2*g).*((h_0^(5/2))-(5*a^2*sqrt(2*g).*t./(2*(tan(pi/2
- theta))^2))).^(1/5)
```

```
v_dot = 1×184
   -0.0617   -0.0616   -0.0616   -0.0615   -0.0614   -0.0614   -0.0613 ⋯
```

```
plot(t,v_dot)
hold on

ylabel('Rate of Volumetric Flow (cubic metre/second')
xlabel('Time (seconds)')
legend('a = 0.03', 'a = 0.06','a = 0.09')
title ['Rate of Volumetric Flow versus Time for Varying a']
hold off
```